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Noncommutative Tachyons and K-Theory

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We show that the relation between D-branes and noncommutative tachyons leads very naturally to the relation between D-branes and K-theory. We also discuss some relations between D-branes and K-homology, provide a noncommutative generalization of the ABS construction, and give a simple physical interpretation of Bott periodicity. In addition, a framework for constructing Neveu-Schwarz fivebranes as noncommutative solitons is proposed.

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1. Introduction

Dbranes can be incorporated into open string field theory as solitons of tachyon configurations [1,2,3,4,5,6] and carry charges which take values in K-theory [7,2,3]. It was recently pointed out [8,9] that the description of D -branes as solitons in the open string tachyon field theory simplifies dramatically when a B field is turned on, thus making the tachyon field theory into a noncommutative field theory with the D -branes appearing as noncommutative solitons [10].

One point of the following paper is that this description provides another point of view on the relation between D -branes and K-theory. Indeed this point of view makes the relation between D -branes and K-theory manifest.

A second, more speculative point we would like to make is the following. In the discussion below we will encounter some simple C^* -algebras. It is natural to wonder if replacing string field algebras by C^* -algebras leads to some new and interesting string backgrounds, or whether the theory of C^* algebras should play a more fundamental role in brane physics.

Some observations closely related to this paper have been independently made in [11,12]. Some of the points below were made in lectures at Strings 2000 [13]. Other recent papers suggesting a role of K-homology in D-brane physics include [14,15,16,17].

2. Noncommutative tachyons are maps to classifying spaces

We first consider noncommutative tachyons in the bosonic string. The basic setup in [9] is that we consider spacetime to be a product $X \times R^{2n}$, where X is a $26 - 2n$ manifold. ([9] take X to be $R^{25-2n,1}$, but the generalization to arbitrary X is easy, and quite important for our point below.)

We now consider open bosonic string field theory with target $X \times R^{2n}$. The action depends on the on-shell background values of the closed string fields $g_{\mu\nu}, g_s, B_{\mu\nu}$, where g_s is the closed string coupling. We take g_s to be small, and assume that the natural generalization of the flat space formulae to curved $g_{\mu\nu}$ applies.

If the tachyon effective action at $B = 0$ is:

$$S = \frac{C}{g_s} \int_{X \times R^2} d^{26}x \sqrt{\det g} \left(\frac{1}{2} f(T) g^{\mu\nu} \partial_\mu T \partial_\nu T - V(T) + \dots \right) \quad (2.1)$$

where C is a constant and T is the tachyon field, then the generalization to $B \neq 0$ is given in terms of a noncommutative field theory [18,19,20]:

$$S = \frac{C}{G_s} \int_{X \times R^2} d^{26}x \sqrt{\det G} \left(\frac{1}{2} f(T) G^{\mu\nu} D_\mu T D_\nu T - V(T) + \dots \right) \quad (2.2)$$

where G_s and $G_{\mu\nu}$ are the open string coupling and metric, given by standard formulae [21,22,20]. The effect of B is to transform $g_s \rightarrow G_s, g_{\mu\nu} \rightarrow G_{\mu\nu}$ and commutative products of fields to noncommutative products taken with the Moyal product. In addition, B induces a non-zero coupling of the tachyon to the noncommutative $U(1)$ gauge field [23].

The tachyon potential is

$$V(T) = V_0 - m^2 T * T + \lambda T * T * T + \dots \quad (2.3)$$

There are also higher derivative terms in (2.2) that we have ignored.

The construction in [8,9] is heavily based on the noncommutative solitons of [10]. According to [10] the most effective way to think about the tachyon dependence on the noncommutative directions is in terms of operators on Hilbert space. The coordinates x^{2i-1}, x^{2i} on the transverse R^{2n} satisfy

$$[x^{2i-1}, x^{2i}] := x^{2i-1} * x^{2i} - x^{2i} * x^{2i-1} = -i\theta_i \quad (2.4)$$

where the θ^i are the skew eigenvalues of the parameter θ^{ij} appearing in the Moyal product.

Letting x^a denote commutative coordinates along X , $x^i, i = 1, \dots, 2n$ denoting the noncommutative coordinates, the tachyon field $T(x^a, x^i)$ is now regarded as an operator valued function of the x^a . What kind of operator can T be? Since T is a real field, T should be a self-adjoint operator. Since we would like to speak of continuous tachyon fields, T should be a map of X into a C^* algebra, and since all such algebras are subalgebras of the algebra of bounded operators on Hilbert space we regard T as a continuous map:

$$T : X \rightarrow \mathcal{B} \quad (2.5)$$

where \mathcal{B} is the C^* -algebra of bounded operators on a Hilbert space \mathcal{H} and we use the norm topology.

In fact, since we wish to have an action, T should have a derivative¹. Moreover, the gauge fields should be introduced using unbounded operators $D_i = \theta_{ij}^{-1} \text{ad} X_j + \text{ad} A_i$ on \mathcal{H} .

¹ More precisely, T should have a Frechét derivative.

After integrating out the massive string fields the effective action for the tachyon and gauge fields takes the form

$$S = \frac{C}{G_s} \int_X d^{26-2n} x \sqrt{\det G} \left(\text{Tr} \left[\frac{1}{2} f(T) G^{ab} D_a T D_b T + \frac{1}{2} f(T) G^{ij} [D_i, T] [T, D_j^\dagger] \right. \right. \\ \left. \left. - V(T) - \frac{1}{4} h(T) G^{ik} G^{jl} F_{ij} F_{kl} - \frac{1}{4} h(T) G^{ac} G^{bd} F_{ab} F_{cd} \right] + \dots \right) \quad (2.6)$$

Here Tr is the trace of the operator on Hilbert space, x^a run over the commuting coordinate directions on X . Evidently, in addition to our other criteria, certain combinations of the map T in (2.5) must be trace class in order to have a finite action.

Let us now consider the limit of [9], $\alpha' B_{ij} \rightarrow \infty$, or equivalently, $\theta^{ij}/\alpha' \rightarrow 0$, and consider constant tachyon field configurations $\partial_a T = 0$. Then by rescaling the coordinates to remove θ_i from the star product one sees that the action reduces to the potential term as $\alpha' B_{ij} \rightarrow \infty$ and hence T must satisfy $V'(T) = 0$.

As noted by [10] this can be solved by

$$T = \sum_i \lambda_i P_i \quad (2.7)$$

where P_i are orthogonal projection operators and λ_i are stationary points for $V(T)$.

In the bosonic string formulated in Witten's open string field theory with $*$ product the potential is purely cubic. If we assume the basic shape of the potential remains unchanged after integrating out massive string fields (recent computations [24,25,6,26] have provided nontrivial evidence that this is correct), then there are two stationary points $\lambda = 0, \lambda = t_*$. If we choose t_* to correspond to the perturbative open bosonic string vacuum, with $V(t_*)$ given by the tension of the D25 brane, then Sen's conjecture states that $V(0) = 0$ represents the closed string vacuum. Therefore, the only nontrivial constant solution to (2.7) is $T = t_* P_n$ where P_n is a rank n projection operator.

Now, in the limit of [9] the action is proportional to $\text{Tr} V(T) = nV(t_*)$ even if the projection operator P_n varies as we move in X . We immediately see the close connection to K -theory. Slowly varying tachyonic field configurations are given by maps from X into the space of rank n projection operators in Hilbert space. This space of projection operators is sometimes denoted $BU(n)$, so we have

$$T : X \rightarrow BU(n) \quad (2.8)$$

If we consider a rank $n < k$ projection operator in the finite dimensional Hilbert space C^k then the space of such projection operators is clearly $U(k)/(U(n) \times U(k-n))$. The space $BU(n)$ is defined as the inductive limit of this quotient space as $k \rightarrow \infty$.

The space $BU(n)$ is topologically intricate, and if X is topologically nontrivial then the set of homotopy classes of maps $[X, BU(n)]$ can be nontrivial. Indeed, $BU(n)$ is a model for a “classifying space” of vector bundles. This means there is an isomorphism

$$Vect_n(X) \cong [X, BU(n)] \quad (2.9)$$

where $Vect_n(X)$ are the isomorphism classes of complex vector bundles on X of rank n . This is explained in detail in [27,28]. In this way we relate homotopy classes of tachyon field configurations directly to isomorphism classes of vector bundles, and therefore to K -theory classes.

In the bosonic string the physical interpretation of these K -theory classes is less clear than in type II theory since the branes carry no conserved charges and presumably are unstable, even if the K -theory class is non-trivial. Our hypothesis is that these K -theory classes label inequivalent unstable D-brane configurations or boundary states of the bosonic string.

It would be very interesting to extend this discussion to the case of finite θ and to include the effects of second derivatives. Such considerations lead to many new questions beyond the scope of this paper. Some of these considerations indicate the relevance of a nonlinear sigma model with target space $BU(n)$.²

3. Witten’s factorization of the open string $*$ product algebra

We now consider spacetime of the form $X \times R^2$ with X a 24-manifold. We also assume the metric factorizes and denote the closed string metric on X by g_{ab} and the closed string metric on R^2 by g_{ij} . Witten has observed in [30] that in the limit of [9], where the closed string metric g_{ij} is fixed and $\alpha' B_{ij} \rightarrow \infty$, (so the open metric $G^{ij} \rightarrow 0$) the $*$ algebra of open string field theory factorizes as $\mathcal{A} \rightarrow \mathcal{A}_0 \otimes \mathcal{A}_1$. Here \mathcal{A}_0 is the algebra of the vertex operators in the 26 dimensional open bosonic string with zero momentum in the noncommutative directions and \mathcal{A}_1 is the algebra of noncommutative functions on R^2 .

² Such sigma models have been considered in a superficially different context by Losev, Nekrasov, and Shatashvili [29].

We can trivially extend the analysis of [30] by considering the following two scaling limits. In the first we take $B_{ij} = tB_{ij}^0$ and $g_{ab} = t^2g_{ab}^0$ and take $t \rightarrow \infty$ keeping B^0 , g_{ab}^0 and g_{ij} fixed. In this limit the string algebra factorizes as above but with \mathcal{A}_0 the algebra of zero momentum vertex operators and $\mathcal{A}_1 = C(X) \otimes C_B(R^2)$ where the first term is the commutative algebra of functions on X and the second is the noncommutative algebra of functions on R^2 defined by the Moyal product.³ The second scaling limit takes $B_{ij} = 0$ and scales both g_{ab} and g_{ij} as t^2 . In this limit the string algebra factorizes with $\mathcal{A}_1 = C(X \times R^2) = C(X) \otimes C(R^2)$ being the algebra of commutative functions on $X \times R^2$.

It is natural to expect that the set of D-branes, or boundary states is somehow connected with a K-theory of the algebra $\mathcal{A}_0 \otimes \mathcal{A}_1$. However, since \mathcal{A}_0 is a vertex operator algebra, the meaning of its K-theory definitely requires some explanation. Without answering this question we can at least ask what we can say without knowing too much about $K(\mathcal{A}_0)$.

Our working hypothesis is that $\mathcal{A}_0, \mathcal{A}_1$ behave similar to C^* algebras. In C^* -algebra theory there is a Kunneth-type theorem which implies that, modulo torsion, we may identify $K(\mathcal{A}_0) \otimes K(\mathcal{A}_1)$ with $K(\mathcal{A}_0 \otimes \mathcal{A}_1)$. (See [31], Theorem 23.1.3.). Therefore, we will focus on the K theory of the algebra \mathcal{A}_1 in the next section.

4. Bott periodicity and noncommutative solitons

The algebra of functions \mathcal{A}_1 is very different for $B = 0$ and for $B \neq 0$. Nevertheless we expect the K -theory classification of branes to be unmodified when we turn on B and scale the metric. We will interpret this statement as a manifestation of Bott periodicity. (See [32] for a related remark.)

Bott periodicity is usually formulated as

$$K(X) \cong K(X \times S^2) = K_{cpt}(X \times R^2) \quad (4.1)$$

In [9] X is R^{23+1} with R^2 as the transverse 2 dimensions to the D23 brane constructed as a noncommutative soliton of the tachyon field theory. Equation (4.1) can be translated into the algebraic setting:

$$K(C(X)) \cong K(C(X) \otimes C_0(R^2)) \quad (4.2)$$

³ There is an important question of whether the functions should be compactly supported, or not. We believe that rapid falloff, or compact support is appropriate.

where $C(X)$ is the algebra of continuous functions on X , and $C_0(R^2)$ is the algebra of continuous functions going to zero at infinity.

K -theory is unchanged under “Morita equivalence.” Therefore:

$$K(C(X)) = K(C(X) \otimes \text{Mat}_N(C)) \quad (4.3)$$

Moreover, the norm-closure of the $N \rightarrow \infty$ limit of $\text{Mat}_N(C)$ is the algebra of compact operators \mathcal{K} . Since K -theory behaves well under inductive limits,

$$K(C(X)) = K(C(X) \otimes \mathcal{K}). \quad (4.4)$$

If the transverse coordinates satisfy $[x^1, x^2] = -i\theta$ (θ is real) then the Stone-von Neuman theorem says there is a unique irreducible unitary representation \mathcal{H} , i.e. the Hilbert space of quantum mechanics. Moreover, to any $f \in \mathcal{S}(R^2)$, the Schwarz space of functions of rapid decrease, the Weyl ordered operators,

$$T(f) = \int dp_1 dp_2 \hat{f}(p_1, p_2) \exp[i(p_1 \hat{x}^1 + p_2 \hat{x}^2)], \quad (4.5)$$

where $\hat{f}(p_1, p_2)$ is the Fourier transform, generate the algebra \mathcal{K} of compact operators [33]. If we suppose that the classification of D -branes is unchanged in the limit $B \rightarrow 0$ then it follows that $K(C(X) \otimes C_B(R^2)) \equiv K(C(X) \otimes \mathcal{K}) = K(C(X) \otimes C_0(R^2))$. Combining this with Morita equivalence we obtain the statement of Bott periodicity.

5. K-theoretic classification of D-branes from tachyons in type IIB Strings

Let us now turn to the tachyon field in the construction of type II D -branes via noncommutative solitons. We will focus on the case of BPS IIB branes. As shown in [30] the tachyon field must satisfy:

$$T\bar{T}T = T \quad (5.1)$$

where \bar{T} is the Hermitian conjugate of T . Equation (5.1) is the defining equation of a “partial isometry.” Moreover, the net brane charge is given by the index of T . In an effective field theory approach ⁴ the tachyon potential has the form

$$V(T, \bar{T}) = U(\bar{T}T - 1) + U(T\bar{T} - 1) \quad (5.2)$$

⁴ This result could presumably also be derived in string field theory

To have a finite energy configuration the kernels of both T and \bar{T} must be finite dimensional, thus T should be both a Fredholm operator and a partial isometry.

Once again, we split spacetime as $X \times R_B^{2n}$, where X has dimension $10 - 2n$ and might be topologically nontrivial. If we consider X -dependent configurations with finite net number of branes then the tachyon field will give us a map

$$T : X \rightarrow \mathcal{F} \quad (5.3)$$

where \mathcal{F} are the Fredholm operators. But this is exactly one model for K-theory! [28][34]. Moreover, the map $[X, \mathcal{F}] \rightarrow K^0(X) \rightarrow 0$ is given by taking the index bundle whose fiber at $x \in X$ is just $Ind(T)_x := Ker(T(x)) - Cok(T(x))$ and we identify this as the K -theory class of the Chan-Paton space of the D-brane. The argument that the map is onto, given in appendix A of [28], shows that there is no loss of generality in supposing that the Fredholm operator is in fact a partial isometry. Thus, one recovers in a very straightforward way the classification of type II D-brane charge in terms of K-theory.

A closely related remark has been made (independently) by Witten in [11] in the type IIA context. Here Witten uses the Fredholm model identifying $K^1(X)$ with $[X, \mathcal{F}^{sa}]$ where \mathcal{F}^{sa} are the self-adjoint Fredholm operators. This model is due to Atiyah and Singer [35].

6. Toeplitz Operators and the ABS Construction

In the explicit solution for the D7 brane as a vortex in the noncommutative plane, explained in [9,30,36] the tachyon operator T is a special kind of partial isometry, namely, a shift operator $T = S$ where S is the shift operator

$$S : |n\rangle \rightarrow |n+1\rangle, \quad n \geq 0 \quad (6.1)$$

in a “harmonic oscillator” basis $|n\rangle$, $n \geq 0$, for a separable Hilbert space. Note that $S^*S = 1$, but SS^* is not 1, indeed, $SS^* = 1 - |0\rangle\langle 0|$. The C^* algebra generated by an operator such that $S^*S = 1$, but $SS^* \neq 1$ is unique, and known as the “Toeplitz algebra.” This algebra can be realized in several ways, and the following is particularly apt for discussing generalizations of noncommutative tachyons.

We consider our Hilbert space to be the Hilbert space of square integrable functions on the circle, $L^2(S^1)$. The functions $\frac{1}{\sqrt{2\pi}}e^{in\theta}$ define a complete orthonormal basis $|n\rangle$ for $n \in \mathbb{Z}$. Given a continuous function $f(\theta)$ we may associate an operator $M_f : \mathcal{H} \rightarrow \mathcal{H}$

simply by multiplying a wavefunction $\psi(\theta)$ by $f(\theta)$. This gives a representation of the commutative C^* algebra $C(S^1)$ on \mathcal{H} . Now consider the Dirac operator $D = -id/d\theta$ and split the Hilbert space into the negative and nonnegative modes of D . Let P be the orthogonal projection onto the positive subspace H_+ of $L^2(S^1)$ spanned by $|n\rangle$ with $n \geq 0$. Equivalently, we could view P as the projection onto the subspace of $L^2(S^1)$ consisting of the boundary values of holomorphic functions. Then given a function $f(\theta)$ on S^1 we can define a Toeplitz operator which maps H_+ to H_+ by $T_f = PM_f$. Note that if f has negative Fourier modes then M_f does not preserve H_+ , and hence the projector P acts nontrivially. For example, if $f_\ell = e^{i\ell\theta}$, then T_{f_ℓ} is just the shift operator S^ℓ for $\ell > 0$, but has a kernel for $\ell < 0$. Quite generally, $(T_f)^\dagger = T_{f^*}$, so $f \rightarrow T_f$ preserves the adjoint $*$ action. However, the map f to T_f is not a homomorphism. Indeed, an easy computation shows that

$$T_1 - T_{f_\ell} T_{f_\ell^*} = P_\ell \quad (6.2)$$

is the projection operator onto the first ℓ levels in H_+ . This is a compact operator, and in general it can be shown that, while $T_f T_g \neq T_{fg}$, the difference $T_f T_g - T_{fg}$ is a compact operator.

In what follows, this construction of Toeplitz operators will be generalized to L^2 functions on odd spheres in order to relate the index of Toeplitz operators to the winding number of ABS configurations.

6.1. Noncommutative ABS Construction

Let us now generalize the construction of T in [9,30] allowing for a $2p$ -dimensional transverse noncommutative space⁵.

First, we construct the noncommutative tachyon field. Let us skew-diagonalize θ^{ij} and take:

$$[x^{2i-1}, x^{2i}] = -i\theta_i \quad \theta_i > 0, i = 1, \dots, p \quad (6.3)$$

Moreover, we consider the irreducible Clifford representation γ_i for \mathcal{Cl}_{2p} . These are $2^p \times 2^p$ dimensional complex Hermitian matrices of the form:

$$\gamma_i = \begin{pmatrix} 0 & \Gamma_i \\ \bar{\Gamma}_i & 0 \end{pmatrix} \quad (6.4)$$

⁵ The construction in [30] includes the possibility of a $2p$ -dimensional transverse space for a single $D9$ -anti $D9$ pair. Here we generalize this to 2^p pairs in order to explain the relation between the index of T and the winding number of the ABS configuration.

Now we take the noncommutative tachyon to be of the same form as the commutative ABS configuration [37,2]:

$$T = f(r)\Gamma_i x^i \quad (6.5)$$

except that we now regard the tachyon as an operator

$$T : \mathcal{H} \otimes S^- \rightarrow \mathcal{H} \otimes S^+ \quad (6.6)$$

where the Hilbert space \mathcal{H} is realized as a representation of p oscillators and S^-, S^+ are negative and positive spin representations. To be specific, we will represent \mathcal{H} as the Bargmann quantization

$$\mathcal{H}_B = Hol(C^p, \exp[-2 \sum \theta_i |z_i|^2] dv) \quad (6.7)$$

The wavefunctions are holomorphic functions of $z_i = x^{2i-1} + ix^{2i}$, normalizable with respect to the above measure and dv is the standard Euclidean volume element. An orthogonal basis for \mathcal{H}_B is provided by the monomials $z^k := \prod_i (z_i)^{k_i}$. We will let k stand for a multiindex $k \in (\mathbb{Z}_+)^p$.

We now show that T is a Fredholm operator, and also show how to determine $f(r)$ from the equations $T\bar{T}T = T$, $\bar{T}T\bar{T} = \bar{T}$. The key calculation is

$$\begin{aligned} \Gamma_i x^i \bar{\Gamma}_j x^j &= \sum_{i=1}^p 2\theta_i (N_i + \frac{1}{2}) - i\Sigma_{ij}\theta^{ij} \\ \bar{\Gamma}_i x^i \Gamma_j x^j &= \sum_{i=1}^p 2\theta_i (N_i + \frac{1}{2}) - i\bar{\Sigma}_{ij}\theta^{ij} \end{aligned} \quad (6.8)$$

Here $\Sigma_{ij} = \frac{1}{4}(\Gamma_i \bar{\Gamma}_j - \Gamma_j \bar{\Gamma}_i)$, $\bar{\Sigma}_{ij} = \frac{1}{4}(\bar{\Gamma}_i \Gamma_j - \bar{\Gamma}_j \Gamma_i)$ and $N_i = a_i^\dagger a_i$ is the i th occupation number. The second terms in (6.8) are diagonalized by the spinor weights to be $\sum_{i=1}^p \pm \theta_i$. Our convention is that in the second equation of (6.8) we have a spinor weight giving $-\sum \theta_i$. Therefore, the first operator has no kernel and the second operator has a one dimensional kernel, given by the oscillator ground state times the lowest weight spinor. Thus,

$$\bar{T} = \bar{\Gamma}_i x^i \frac{1}{\sqrt{\Gamma_i x^i \bar{\Gamma}_i x^i}} \quad (6.9)$$

satisfies the equation $T\bar{T}T = T$, has no kernel and is of index -1 . We will refer to this as the “noncommutative ABS construction.” In order to explain the relation to the ABS

construction we would like to make sense of restricting the tachyon field to a sphere in the noncommutative space. Classically, we restrict the field T to the solutions of the equation

$$\sum_i |z_i|^2 = R^2 \quad (6.10)$$

defining the sphere Σ of dimension $2p - 1$ and radius R . At nonzero B field the z_i become noncommuting, so the question arises as to what it means to restrict the operator to a noncommutative sphere. We will now propose one interpretation of what this might mean.

In quantum mechanics, restricting the wavefunctions in the Bargman space \mathcal{H}_B to the sphere produces the Hardy space \mathcal{H}_Σ . This is the Hilbert subspace of $L^2(\Sigma; d\Omega)$ defined by the boundary values of holomorphic functions. Here $d\Omega$ is the standard round measure on the sphere such that $dv = R^{2p-1} dR d\Omega$. The projection operator from L^2 to \mathcal{H}_Σ is given by

$$(Pf)(z) = \int_\Sigma K_\Sigma(z, w) f(w) d\Omega \quad (6.11)$$

$$K_\Sigma(z, w) = (1 - z \cdot \bar{w})^{-p}$$

An orthogonal basis for the Hardy space is again given by $\varphi_k = z^k$. Note, that the norm of these states in the Hardy space is

$$(z^k, z^{k'}) = \delta_{k,k'} \frac{2\pi^p \prod_i (k_i)!}{\Gamma(|k| + p)}$$

where $|k| = \sum k_i$ for a multi-index k .

Now let us consider the action of classical coordinates z_i, \bar{z}_i on the Hardy space \mathcal{H}_Σ . To make sense of this we need to define Toeplitz operators. In general, if $f : \Sigma \rightarrow \mathbb{C}$ is any function we define $\mathcal{T}_f := PM_f$ where $M_f : \mathcal{H}_\Sigma \rightarrow L^2$ is the operator of multiplication by f . The operators $\mathcal{T}_{z_i}, \mathcal{T}_{\bar{z}_i}$ are easily computed:

$$\begin{aligned} \mathcal{T}_{z_i} \varphi_k &= \varphi_{k+e_i} \\ \mathcal{T}_{\bar{z}_i} \varphi_k &= 0 \quad \text{if } k_i = 0 \\ &= 2\pi \frac{k_i}{|k| + p - 1} \varphi_{k-e_i} \quad \text{if } k_i > 0 \end{aligned} \quad (6.12)$$

where e_i is the i^{th} unit vector in $(\mathbb{Z}_+)^p$.

By considering the Hilbert space $\mathcal{H}_\Sigma \otimes \mathbb{C}^N$, Toeplitz operators for functions are easily generalized to Toeplitz operators for matrix valued functions $f : \Sigma \rightarrow Mat_N(\mathbb{C})$, and hence we can consider our tachyon operator (6.6) above as a Toeplitz operator

$$T : \mathcal{H}_\Sigma \otimes S \rightarrow \mathcal{H}_\Sigma \otimes S \quad (6.13)$$

where $S^+ \cong S^- \cong S$ is the irreducible spin representation in odd dimensions. The Toeplitz operator is the projection P_+ composed with matrix multiplication by $\beta : \Sigma \rightarrow GL(N, C)$ given, essentially by the ABS construction:

$$\beta(x) = \Gamma_i x^i \frac{1}{\sqrt{x^i x^i + \text{const.}}}$$

The operator T in (6.13) is bounded and Fredholm. Now, although the restriction map $\mathcal{H}_B \rightarrow \mathcal{H}_\Sigma$ is *not* unitary it is 1-1 and onto. Therefore, the *index* of T on \mathcal{H}_B will be the same as the index of T on \mathcal{H}_Σ .

Now, we can invoke the index theorem of Boutet de Monvel [38], according to which the index is:

$$\text{Index}(T|_{\mathcal{H}_\Sigma}) = \int_\Sigma \text{ch}(\beta) Td(T\Sigma) \quad (6.14)$$

Here

$$\text{ch}(\beta) = \beta^* \left(\sum_{j \geq 0} (-1)^{j-1} \frac{\omega_{2j-1}}{(j-1)!} \right)$$

and ω_i are standard generators of $H^i(GL(N, C), Q)$. Since $Td(T\Sigma) = 1$ in this case we have a direct connection between the index of the tachyon operator on \mathcal{H}_B , and the winding number of the classical ABS tachyon.

7. Remarks on the relation to K-homology

The noncommutative ABS construction in the previous section leads rather naturally to a relation between D-branes and the work of Brown, Douglas, and Fillmore (BDF) on the classification of algebras of essentially normal operators [39]. In this section we will give a brief review of that work, and then explain the relation to D-branes.

7.1. Brief review of BDF

Expository discussion of [39] can be found in [40,31,41,42,43]. For the readers' convenience we give a brief summary here.

In Matrix theory [44], spacetime emerges from an algebra of commuting operators. Here we will discuss algebras of “almost commuting” operators in the belief that these are related to D-branes. Recall that by Gelfand’s theorem, C^* algebras of commuting operators are naturally associated to Hausdorff topological spaces X by considering the algebra of

continuous functions $C(X)$.⁶ Isomorphism classes of algebras are in 1-1 correspondence with homeomorphism classes of spaces. We will now consider noncommutative C^* algebras \mathcal{A} which fit into the short exact sequence:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \xrightarrow{\beta} C(X) \rightarrow 0 \quad (7.1)$$

for some fixed space X . Note that if T_f denotes some operator in \mathcal{A} mapping to the function f under β , then $T_{f_1}T_{f_2} - T_{f_1f_2}$ is in the kernel of β , and hence is a compact operator. It follows that $[T_{f_1}, T_{f_2}]$ is compact and thus the algebra \mathcal{A} is thus “almost commuting” in the sense that compact operators are considered to be “small.” An example of such an extension is given by the Toeplitz algebra generated by the shift operators, described at the beginning of section 6: $S = T_z \rightarrow z$ defines a C^* morphism onto the continuous functions on $X = S^1$.

In [39] BDF investigated extensions of the form (7.1) for fixed X . To any such extension we can associate a C^* -algebra morphism (called the “Busby invariant”) $\tau : C(X) \rightarrow Q(\mathcal{H})$ where Q is the “Calkin algebra” defined by $Q(\mathcal{H}) := B(\mathcal{H})/\mathcal{K}$ where $B(\mathcal{H})$ is the algebra of bounded operators on a separable Hilbert space. Indeed, for any $f \in C(X)$ we choose an operator $T_f \in \mathcal{A}$ projecting to it, and define τ by: $\tau(f) = \pi(T_f)$ where $\pi : B(\mathcal{H}) \rightarrow Q(\mathcal{H})$ is the projection. Since $T_{f_1}T_{f_2} - T_{f_1f_2}$ is a compact operator, τ is an algebra homomorphism. Conversely, given a C^* -algebra morphism $\tau : C(X) \rightarrow Q(\mathcal{H})$ one can form an extension (7.1), and, up to a natural notion of isomorphism, τ uniquely characterizes the extension. Full details can be found in ch. 3 of [42]. Suffice it to say here that, given $\tau : C(X) \rightarrow Q(\mathcal{H})$ we can form

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A}' \rightarrow C(X) \rightarrow 0 \quad (7.2)$$

by defining

$$\mathcal{A}' := \{(\mathcal{O}, f) : \pi(\mathcal{O}) = \tau(f)\} \subset B(\mathcal{H}) \oplus C(X) \quad (7.3)$$

and that (7.2) is equivalent to (7.1) in the sense that there is an isomorphism $\psi : \mathcal{A} \rightarrow \mathcal{A}'$ compatible with the two sequences.

One of the reasons the Busby invariant is useful is that it allows one to define a notion of direct sum of extensions. In order to do this we must first introduce “unitary

⁶ If X is noncompact, we add the condition that $f \rightarrow 0$ at infinity, and correspondingly $C(X)$ does not have a unit. For simplicity of discussion, we henceforth assume X is compact in this subsection.

equivalence,” also known as “strong equivalence.” Two extensions (7.1) are “strongly equivalent” if there is a unitary operator U on \mathcal{H} such that the Busby invariants are related by $\tau_2(f) = \pi(U)\tau_1(f)\pi(U)^*$. Let $\mathbf{Ext}(C(X), \mathcal{K})$ denote the set of strong equivalence classes of extensions of $C(X)$ by \mathcal{K} . A direct sum operation on $\mathbf{Ext}(C(X), \mathcal{K})$ can then be defined by taking the extension corresponding to the Busby invariant

$$\tau_1 \oplus \tau_2 : C(X) \rightarrow Q(\mathcal{H}) \oplus Q(\mathcal{H}) \rightarrow Q(\mathcal{H} \oplus \mathcal{H}) \cong Q(\mathcal{H}). \quad (7.4)$$

It turns out that (7.4) defines a semigroup operation on $\mathbf{Ext}(C(X), \mathcal{K})$. Thus far, the theory could have been developed for general extensions $\mathbf{Ext}(A_1, A_2)$ of arbitrary C^* algebras A_1 by A_2 . However, specializing to $A_1 = C(X)$ and $A_2 = \mathcal{K}$, a number of nice things begin to happen. It turns out that there is a natural zero in the semigroup, corresponding to the “trivial extensions.” These are extensions for which the Busby invariant lifts to $B(\mathcal{H})$; equivalently, they are extensions such that the sequence (7.1) splits, and hence we can unambiguously write every operator in \mathcal{A} in the form $T_f = N_f + k$ with $k \in \mathcal{K}$ and $N_{f_1}N_{f_2} = N_{f_1 f_2}$. Let $Ext(C(X), \mathcal{K})$ be the quotient of $\mathbf{Ext}(C(X), \mathcal{K})$ by the trivial extensions. In the above references it is shown that every extension has an inverse (up to the addition of a trivial extension) so that $Ext(C(X), \mathcal{K})$ in fact is an abelian group. Moreover, $Ext(C(X), \mathcal{K})$ can even be used to define a homology theory on X ! Indeed, if we define

$$\begin{aligned} K_1^a(X) &= Ext(C(X), \mathcal{K}) \\ K_0^a(X) &= Ext(C(X) \otimes C_0(\mathbb{R}), \mathcal{K}) \end{aligned} \quad (7.5)$$

(the superscript “a” is for “analytic,” K-homology, as opposed to “topological” K-homology), then it turns out that $K_*^a(X)$ is a mod 2 periodic homology theory, dual to K -theory.

One way to make the relation to a homology theory more evident is to introduce the noncommutative spheres $\mathcal{S}^0, \mathcal{S}^1$ with function spaces

$$\begin{aligned} C(\mathcal{S}^1) &= \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in B(\mathcal{H}), a_{12}, a_{21} \in \mathcal{K} \right\} \\ C(\mathcal{S}^0) &= \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} : a_{11} - a_{22} \in \mathcal{K} \right\} \end{aligned} \quad (7.6)$$

and then define $K_i(X)$ to be homotopy classes of maps of X into \mathcal{S}^i , $K_i(X) := [\mathcal{S}^i, X]$. In the noncommutative setting this amounts to the homotopy classes of $*$ -homomorphisms $C(X) \rightarrow C(\mathcal{S}^i)$. (The equivalence of this definition to what we described above is hardly obvious. The necessary technical details can be found in [31], sections 15.7 and 15.8.)

7.2. Algebra extensions associated to IIA branes

We will now review a construction from [41] which may be interpreted as saying that every IIA D-brane naturally provides a nontrivial extension of the algebra of functions on spacetime by compact operators.

Let W be an odd-dimensional $Spin^c$ submanifold of a spacetime X . W is equipped with a complex vector bundle E with connection and inherits a metric from X . We think of W as the IIA brane worldvolume and E as its Chan-Paton bundle. Using the above data we can form the Hilbert space \mathcal{H} of L^2 spinors with values in $S \otimes E$, where $S \rightarrow W$ is the spin bundle. Denote the Dirac operator on $S \otimes E$ by \not{D}_E . Assuming the connection and metric are generic, \not{D}_E will have no zeromodes and we can decompose the Hilbert space into the positive and negative eigenspaces of \not{D}_E : $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The commutative algebra $C(W)$ is represented on \mathcal{H} by multiplication operators M_f for $f \in C(W)$. In general, M_f does not preserve the subspace \mathcal{H}_+ , but if we take the “compression” of M_f by composing with the projection operator $P_+ : \mathcal{H} \rightarrow \mathcal{H}_+$ then we can define a Toeplitz operator $T_f = P_+ M_f : \mathcal{H}_+ \rightarrow \mathcal{H}_+$. As in the case $W = S^1$ described previously, it turns out that $T_{f_1} T_{f_2} - T_{f_1 f_2}$ is a compact operator and we obtain an extension

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \rightarrow C(W) \rightarrow 0 \quad (7.7)$$

where \mathcal{A} is the C^* algebra generated by the T_f . By using pullback $\phi^* : C(X) \rightarrow C(W)$ we obtain an extension of the algebra of functions on all of spacetime. That is, if $\phi : W \rightarrow X$ is a continuous map then we can define a Busby invariant $\tau\phi^* : C(X) \rightarrow Q(\mathcal{H})$ from which we get an extension:

$$0 \rightarrow \mathcal{K} \rightarrow \tilde{\mathcal{A}} \rightarrow C(X) \rightarrow 0 \quad (7.8)$$

It is shown in [41] that all classes in $K_1^a(X) = Ext(C(X), \mathcal{K})$ can be obtained from the above construction using a suitable triplet (W, E, ϕ) . Moreover, if a suitable equivalence relation is put on (W, E, ϕ) then classes in $K_1^a(X)$ are in 1-1 correspondence with classes $[(W, E, \phi)]$. The equivalence relations on (W, E, ϕ) make good physical sense: they include cobordism (i.e. continuous deformation of the worldvolume and Chan-Paton bundle) and a natural identification of direct sums of Chan-Paton bundles. In addition they include “vector bundle modification,” a mathematical construction reminiscent of the Myers dielectric effect [45].

It is interesting to compare $K_1^a(X)$ with the group of D-brane charges, thought to be given by $K^1(X)$. If X is compact, even dimensional and spin then, modulo torsion, $K_1(X)$

is isomorphic to $K^1(X)$ by Poincaré duality. However, when we include torsion a puzzling difference emerges. There is a universal coefficient theorem ([31], Theorem 16.3.3):

$$0 \rightarrow \text{Ext}(K^0(X), \mathbb{Z}) \rightarrow \text{Ext}(C(X), \mathbb{Z}) \rightarrow \text{Hom}(K^1(X), \mathbb{Z}) \rightarrow 0 \quad (7.9)$$

Moreover, the sequence splits, so that the torsion can in principle differ from that of $K^1(X)$. This possible discrepancy in torsion charges deserves to be more thoroughly investigated.

7.3. The index theorem

We can now put the noncommutative ABS tachyon field of the previous section into its proper mathematical context: The equivalence of IIB D-brane charges in the commutative and noncommutative theory is simply the equality of the analytic and topological index, expressed in the framework of K -homology (as explained in [41]).

In the language of Brown-Douglas-Filmore, the Toeplitz operators on the Hardy space defines an analytic K -homology class

$$[(\mathcal{H}_\Sigma, \tau)] \in K_{1,a}(\Sigma^{2p-1}) \quad (7.10)$$

where τ is the Busby invariant. That is, the inverse image under $\pi : B(\mathcal{H}) \rightarrow Q(\mathcal{H})$ of $\tau(C(\Sigma^{2p-1}))$ in $B(\mathcal{H}_\Sigma)$ defines an algebra of operators \mathcal{T} providing a nontrivial extension by compact operators:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow C(\Sigma^{2p-1}) \rightarrow 0 \quad (7.11)$$

It is explained in [41] that the K -homology class (7.10) is the same as that determined by the Dirac operator $[\not{D}]$ on Σ using the construction of section 7.2. In particular, the index theorem of Boutet de Monvel follows from the ordinary index theorem.

One usually associates IIB D-brane charge to $K^0(X)$, or for an infinitely extended D-brane with transverse space X_t , to $K_{cpt}^0(X_t)$. The relation to (7.10) is explained as follows. We consider the exact sequence in K -homology for the pair (D^{2p}, Σ^{2p-1}) , where D^{2p} is the disk of dimension $2p$ with boundary Σ^{2p-1} . The connecting homomorphism gives an isomorphism

$$\delta : K_0(D^{2p}, \Sigma^{2p-1}) \cong K_1(\Sigma^{2p-1}) \quad (7.12)$$

In this way, the above construction associates an element of analytic K -homology $K_{0,a}$ to the noncommutative tachyon. By Poincaré duality $K_0 \cong K^0$, (again, up to torsion) and we produce the same K -theory class we expected to associate to a IIB brane.

7.4. Speculations on noncommutative D-branes

The above considerations lead to the idea that it might be fruitful to relax the equivalence relations we have put on the extensions (7.8). As we have discussed, any “commutative D-brane” defines a triple (W, E, ϕ) and hence a particular extension. Conversely, given an abstract extension (7.8) could one extract the data of a D-brane? We can easily answer one simple question about such generalized D-branes, namely: “*Where* is the brane?” as follows. The kernel of the Busby invariant $\tau : C(X) \rightarrow Q$ defines an ideal, and from the Gelfand correspondence therefore defines a subspace $W \subset X$. Concretely, the ideal is the subalgebra of functions vanishing on W . It would be natural to identify W with the worldvolume of a D-brane. Whether or not one can usefully recover other aspects of the structure of a D-brane, and in particular whether extensions (7.8) which do *not* come from triples (W, E, ϕ) can be usefully identified with “noncommutative D-branes” remains an interesting open question.

In any case, inspired by the result of BDF we would like to define an action whose solutions could be considered to be the set of possible IIA D-branes, generalized in the above sense. The action has some interesting similarities to the IKKT action. On the other hand, we caution the reader at the outset that it remains to be seen if the following action will play any useful role in the computation of any physical quantities.

The action is a function of pairs (\mathcal{A}, ϕ) , where \mathcal{A} is a C^* algebra and ϕ is a C^* -algebra morphism $\phi : \mathcal{A} \rightarrow C(X) \rightarrow 0$, and is given by

$$S(\mathcal{A}, \phi) := \sup_{f_1, f_2 \in C(X)} \inf_{\phi(a_i) = f_i} \text{Tr}_D([a_1, a_2][a_1, a_2]^\dagger). \quad (7.13)$$

Here we first take the infimum over all lifts T_f of a pair of functions f . Moreover, Tr_D is the Dixmier trace. Roughly speaking, Tr_D is defined as follows. Let $\mu_n(T)$ be the eigenvalues of $\sqrt{T^\dagger T}$ arranged in decreasing order. Define

$$\text{Tr}_D(T) := \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \mu_n(T). \quad (7.14)$$

For the real story, consult the book by Connes [46].

The action (7.13) is positive semidefinite. So, in any reasonable “space of (\mathcal{A}, ϕ) ” the zeroes of the action are automatically stationary points of minimal action. The action is identically zero only when, for all $f_1, f_2 \in C(X)$ there are lifts T_{f_1}, T_{f_2} such that the commutator $[T_{f_1}, T_{f_2}]$ has singular values falling off faster than $1/\sqrt{n}$. We may expect

the relations (6.12) to give a good approximation to the general behavior of $[T_{f_1}, T_{f_2}]$ on spinors of large energy, and from this we expect that the extensions associated to (W, E, ϕ) described above will be zeroes of the action. Conversely, any zero of the action can be used to define an extension of $C(X)$ by compact operators.

It is interesting to compare the action (7.13) with the IKKT model:

$$S = g_{IK} g_{JL} \text{Tr} \left([X^I, X^J] [X^K, X^L] \right) \quad (7.15)$$

where X^I are $N \times N$ Hermitian matrices and g_{IJ} is a nondegenerate constant metric on \mathbb{R}^{10} . If we consider the X^I as generators of the algebra of functions on \mathbb{R}^9 then there is a certain similarity between (7.13) and (7.15). However we note that

1. The IKKT action does not generalize easily to curved spaces. Even on \mathbb{R}^9 if we attempt to include curved metrics g_{IJ} we run into ordering problems. (See [47], for the state of the art on this problem.)

2. When producing D-branes from Matrix theory the solutions have infinite action. Of course, this is physically appropriate for infinitely extended planar branes. Nevertheless, it would be nice to work with finite action quantities when considering compact branes.

8. Nonzero H -fields and 5-branes

In this section we will focus on a description of Neveu-Schwarz fivebranes in the framework of [9]. We should first discuss what we mean by an NS fivebrane in open string theory. In the original description [48] NS fivebranes are solutions to the closed string equations of motion with topology $M \times S^3 \times R$ with M the fivebrane world volume such that $\int_{S^3} H = Q_5$, H being the NS three-form field and Q_5 the quantized fivebrane charge. Since the tension scales like $1/g_s^2$ with g_s the closed string coupling, these are properly thought of as solitons in the closed string sector rather than the open string sector of the theory where soliton energies scale as $1/g_s$ (as for D-branes). In open string theory we cannot expect to see the detailed form of the closed string solution since closed string states only arise at the loop level in open string theory. We thus define a fivebrane to be a configuration in a ten-dimensional spacetime X with $H \in H^3(X, \mathbb{Z})$ a non-trivial integer class.

Given the scaling of the fivebrane tension with g_s , the close connection between the framework of [9] and Matrix theory [49], and the well-known difficulties in describing fivebranes in Matrix theory [50,51], we can anticipate some problems here as well.

To explain the basic idea and the difficulty one expects, consider taking $X = W \times R_B^2$ to be the world-volume of an unstable $D9$ -brane in IIA with a large B-field on the R^2 component and take $W = R^5 \times S^3$. W represents the commutative part of the $D9$ -brane world volume. The effective action (2.6) contains gauge fields with gauge group $U(\mathcal{H})$ coupled to the tachyon field in the adjoint representation. Since the $U(1)$ component of $U(\mathcal{H})$ (with A_μ proportional to the identity operator) does not couple to T and has infinite action if its field strength is non-zero, it is more correct to say that the gauge group is $PU(\mathcal{H}) \equiv U(\mathcal{H})/U(1)$. Defining more precisely what is meant by the $U(1)$ component when there is non-trivial topology is quite subtle as will be discussed below.

Since $U(\mathcal{H})$ is contractible [52], $\pi_2(PU(\mathcal{H})) = \pi_1(U(1)) = \mathbb{Z}$. Thus we can construct an “instanton” configuration of the $PU(\mathcal{H})$ gauge fields on S^3 by patching together gauge fields on the northern and southern hemispheres using a non-trivial element of $\pi_2(PU(\mathcal{H}))$ on the S^2 equator. Our proposal is that such a twisted $PU(\mathcal{H})$ bundle with the tachyon field $T = t_*$ represents a $D9$ -brane in the presence of a NS fivebrane while condensing the tachyon field to $T = 0$ removes the $D9$ -brane and leaves an NS fivebrane. More generally, we can use non-trivial projection operators for the tachyon to study lower D -branes in the presence of NS fivebranes.

We can now see one difficulty we expect to encounter. Since the NS fivebrane world volume is six-dimensional, it must span $R^5 \in W$ and as well have one component in the noncommutative plane R_B^2 . As a result, the trace in (2.6) is expected to diverge, i.e. the gauge field fieldstrength-squared for the twisted $PU(\mathcal{H})$ connection is not expected to be trace class. More precisely, we expect that if we cut off the trace by summing over a finite number of modes then the trace will diverge in the mode number cutoff. In fact, an evaluation of the gauge action $\int_W \text{Tr} F \wedge *F$ for some examples of smooth nontrivial $PU(\mathcal{H})$ connections shows that the action is in indeed infinite. A proper interpretation of this infinity must be addressed in future work. Here we simply note that since the mode-number cutoff can be interpreted as an infrared cutoff in the noncommutative directions, there is room for an interpretation of the infinite gauge kinetic action as the volume divergence due to the extension of the 5-brane worldvolume in the noncommutative directions.

To see the connection to the fivebrane definition in terms of H , we note that a standard theorem states that principal $PU(\mathcal{H})$ bundles are classified by the Dixmier-Douady class $h \in H^3(W, \mathbb{Z})$.⁷ This class has been interpreted in [53,54,11] as the cohomology class of

⁷ A quick homotopy-theoretic proof is that $BPU(\mathcal{H}) \sim K(\mathbb{Z}, 3)$ since $\Omega BPU(\mathcal{H}) \sim PU(\mathcal{H}) \sim K(\mathbb{Z}, 2) \sim \Omega K(\mathbb{Z}, 3)$.

the H -field of string theory. In addition to the arguments presented in these papers we would like to point out that the reasoning described by Kapustin in [55] for the case when h is torsion in fact can be extended to the case of h non-torsion. This follows because a nontrivial $PU(\mathcal{H})$ bundle defines a nontrivial “bundle gerbe with connection,” (where we are using the terminology explained in [56][57]). Then, using the equivalence to the formulation of Brylinski [58] one can argue that the “holonomy of the $PU(\mathcal{H})$ connection in the fundamental representation” can be given a concrete definition in terms of a covariantly constant section of a line bundle with connection over loop space LW . The line bundle with connection over LW is constructed using the bundle gerbe associated to the $PU(\mathcal{H})$ bundle with connection A in a way explained in [58][56][57].

In more physical terms, we wish to make sense of the expression

$$\exp \left[i \int_D B \right] \text{Tr}_{\mathcal{H}} P \exp \int_{\gamma} A \quad (8.1)$$

in the open string path integral, where D is the disk worldsheet with boundary $\gamma \subset W$, B is the background B -field, and A is a $PU(\mathcal{H})$ connection. In order to define this we must lift A to a compatible $U(\mathcal{H})$ connection \tilde{A} . In so doing the field strength acquires a “ $U(1)$ component” which we denote by $\text{Tr} \tilde{F}$, although since we are working with operators not necessarily of trace class this notation should be handled with great care. The essential physical point is that in infinite dimensions the commutator of two Hermitian operators can be proportional to the identity matrix, the standard example being a pair of operators representing the Heisenberg relations. Consequently the commutator term in $\tilde{F} = d\tilde{A} + \tilde{A}^2$ can in fact contribute to the $U(1)$ component of \tilde{F} and in topologically interesting situations it must do so. This in turn means that the Bianchi identity for the $U(1)$ part of \tilde{F} is *not* $d\text{Tr} \tilde{F} = 0$ but rather $d\text{Tr} \tilde{F} = K$ where K is a globally well-defined 3-form on W . Moreover, by the general results of [58] it follows that the cohomology class of $K/(2\pi i)$ coincides with the Dixmier-Douady class h . Defining the holonomy of A as a covariantly constant section of a bundle over loop space one can follow the strategy of [55] and conclude that the Dixmier-Douady class h must be identified with that of the physical H -field. It is not necessary to assume that h is a torsion class, although in infinite dimensions the trace Tr isolating the $U(1)$ part of the field strength requires an ad-hoc definition [58].

One simple example of a nontrivial $PU(\mathcal{H})$ connection illustrating some of the above general remarks is the following. (This example is a paraphrase of section 4.3 of [58].) We will take the base space to be the three-manifold $S^2 \times S^1$, more appropriate to an

H -monopole. A similar (but more elaborate) example applies directly to S^3 and can be extracted from [57].

We will construct a $PU(\mathcal{H})$ bundle over $S^2 \times S^1$ by starting with a $U(1) \times \mathbb{Z}$ bundle over $S^2 \times S^1$ and then embedding the $U(1) \times \mathbb{Z}$ transition functions into $PU(\mathcal{H})$. The $U(1) \times \mathbb{Z}$ bundle over $S^2 \times S^1$ will simply be $S^3 \times \mathbb{R}$ with a rightaction by $U(1) \times \mathbb{Z}$ given by:

$$\begin{aligned} (u, x) &\sim (ue^{i\chi\sigma^3/2}, x) \\ (u, x) &\sim (u, x+1) \end{aligned} \tag{8.2}$$

Here $u \in S^3$ is identified with an $SU(2)$ matrix, the first line is the right $U(1)$ action and the second line is the \mathbb{Z} action on $x \in \mathbb{R}$. Note that the S^3 is *not* to be thought of as embedded in spacetime. Rather, $W = \mathbb{R}^5 \times S^2 \times S^1$.

We now consider the Heisenberg algebra generated by operators $\hat{\theta}$, and \hat{N} acting on functions in $L^2(S^1)$. This S^1 should be thought of as the fiber in the Hopf fibration $S^3 \rightarrow S^2$. Let $\hat{\theta}$ be the position operator and \hat{N} the integrally-quantized angular momentum, so that $[\hat{\theta}, \hat{N}] = i$. Using these operators we can form a representation of $U(1) \times \mathbb{Z}$ in $PU(\mathcal{H})$ via:

$$(e^{i\chi}, n) \rightarrow e^{in\hat{\theta}} e^{i\chi\hat{N}} \tag{8.3}$$

Note that $e^{in\hat{\theta}}$ and $e^{i\chi\hat{N}}$ commute up to the phase $e^{in\chi}$ and hence (8.3) is indeed a representation of the commutative group $U(1) \times \mathbb{Z}$ in $PU(\mathcal{H})$. Using (8.3) we convert the transition functions of the $U(1) \times \mathbb{Z}$ bundle $S^3 \times \mathbb{R} \rightarrow S^2 \times S^1$ into $PU(\mathcal{H})$ transition functions. Of course, we can (by construction) lift the transition functions to $U(\mathcal{H})$ over contractible open sets in a good cover of $S^2 \times S^1$, but then they will fail to satisfy the cocycle condition.

We now discuss how to isolate the $u(1)$ part. Technically, this is defined in [58] by the splitting of an exact sequence of bundles of the adjoint representation. Here the relevant Lie algebra of operators is generated by the invariant elements $(\hat{N} - x\mathbf{1})$ and $\mathbf{1}$. Note that \mathbb{Z} , being discrete, has no Lie algebra. Therefore, we do not need to include $\hat{\theta}$. Note too that we are forced to choose the combination $(\hat{N} - x\mathbf{1})$ so that $x \sim x+1$ is equivalent to conjugation by $e^{i\hat{\theta}}$. We define “the $u(1)$ part” to be the coefficient of $\mathbf{1}$ in this basis.

As an example of a nontrivial $PU(\mathcal{H})$ connection we choose standard Euler angle coordinates (ϕ, θ, ψ) for $u \in S^3$ and x on \mathbb{R} . Then we may define the connection using the globally defined Lie algebra valued form on $S^3 \times \mathbb{R}$ given by:

$$\begin{aligned} A_+(u, x) &= i\left(d\psi_+ + \frac{1}{2}(1 - \cos\theta)d\phi\right)(\hat{N} - x\mathbf{1}) \\ A_-(u, x) &= i\left(d\psi_- - \frac{1}{2}(1 + \cos\theta)d\phi\right)(\hat{N} - x\mathbf{1}) \end{aligned} \tag{8.4}$$

where we have divided S^3 into two hemispheres labelled by \pm . One may easily check that $A(u, x+1) = e^{i\hat{\theta}} A(u, x) e^{-i\hat{\theta}}$ so this defines a connection on a bundle over $S^2 \times S^1$. According to our definition of the $U(1)$ part of F we have $\text{Tr}(F) = -iAdx$. This is globally defined on $S^3 \times S^1$ but is not basic. It is also not closed, as promised, but $K = d\text{Tr}(F) \wedge dx = -\frac{i}{2} \sin \theta d\theta d\phi dx$ is a basic form, giving the globally defined “gerbe curvature 3-form” on $S^2 \times S^1$ with $\int_{S^2 \times S^1} K/(2\pi i) = 1$.

In view of the above, we believe that by allowing for twisted $PU(\mathcal{H})$ bundles in the formalism of [9] we are able to include the effects of NS 5-branes in the picture of [9]. Indeed, if $A \rightarrow W$ is a twisted bundle with fiber given by \mathcal{K} and Dixmier-Douady class h then $\Gamma(A)$ is an algebra whose (Grothendieck group of) finitely generated projective modules define $K_H(W)$. In the limit of large noncommutativity the tachyon field still defines a projection operator, hence a projective module for this algebra. In the context of type II strings it is important to note that since $PU(\mathcal{H})$ also acts on Fredholm operators, there is also a Fredholm model for $K_H(W)$.

Obviously, many details need to be worked out in the above proposal. We hope to report on this elsewhere.

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